NUMERICAL SOLUTION OF FRACTIONAL-ORDER DIFFERENTIAL EQUATIONS USING THE ADAMS–BASHFORTH–MOULTON (ABM) METHOD

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Abstract

This paper investigates the Adams–Bashforth–Moulton predictor–corrector method for Caputo fractional initial-value problems. The method is formulated using the equivalent Volterra integral equation and product integration on both uniform and nonuniform meshes. The paper presents the classical convergence result together with its precise regularity assumptions, emphasizing that these conditions concern an auxiliary function rather than the solution itself. The mechanism responsible for the limitation of the convergence order is also discussed. In addition, a perturbation stability estimate is established using a weakly singular discrete Gronwall inequality.

Several numerical experiments are presented to illustrate the theoretical results. The experiments include a nonlinear manufactured problem and a control problem with the same exact solution but a right-hand side independent of the fractional parameter. The results demonstrate the expected convergence behavior of the method. Furthermore, the use of a graded mesh improves the convergence properties and significantly reduces the numerical error near the initial point.

Keywords

fractional differential equation, Caputo derivative, Adams–Bashforth–Moulton method, predictor–corrector scheme, product integration, convergence order, graded mesh, stability.

Fractional differential equations (FDEs) replace integer-order derivatives by derivatives of non-integer order. Their nonlocal character makes numerical solution substantially different from the ODE case, because the value at a new time level depends on the whole previous history. Product-integration rules and fractional linear multistep methods therefore play a central role in practical computation [3,6]. For Caputo initial-value problems, the predictor–corrector algorithm of Diethelm, Ford and Freed [1] is a standard approach: it generalises the classical Adams–Bashforth–Moulton construction and treats nonlinear right-hand sides conveniently. Its error analysis is given in [2]. That analysis shows that the convergence order depends on the regularity of the Caputo derivative of the solution, and that initial singular behaviour can reduce the order attained on a uniform mesh; graded meshes have been analysed as a remedy [5,9]. Adams-type error analysis has also been extended to other fractional operators [10], and further variants of the fractional Adams method continue to be studied [7,8].

(a) The ABM weights are derived from the Volterra equation for both uniform and arbitrary (in particular graded) meshes, including a numerically stable way of evaluating the nonuniform weights.

(b) The classical convergence theorem is stated with its precise hypothesis, and it is shown by an explicit calculation which of the tests used here satisfy it.

(c) A pair of controlled experiments isolates the predictor as the cause of the order $1+\alpha$: two problems with the same exact solution and the same g are solved, one with y-dependent and one with y-independent right-hand side.

(d) A test with a genuinely non-smooth g compares uniform and graded meshes with the ABM scheme.

(e) A weakly singular discrete Gronwall argument gives perturbation stability, and the $O(N^2)$ cost is stated. Only scalar equations with $0<\alpha<1$ and a single correction step are considered.

Problem formulation. Consider the nonlinear fractional initial-value problem

$$ {}^{C}D_{0+}^{\alpha} y(t) = f(t, y(t)), \quad 0 < t \le T, \qquad y(0) = y_0, \quad 0 < \alpha < 1. \tag{1} $$

For a sufficiently regular y, the Caputo derivative of order α is

$$ {}^{C}D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} y'(\tau) d\tau. \tag{2} $$

Problem (1) is equivalent to the Volterra integral equation

$$ y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, y(\tau)) d\tau, \tag{3} $$

which is the starting point of the ABM construction. The kernel is weakly singular for $0 < \alpha < 1$, but this singularity is integrated analytically in the product-integration weights below. It should therefore not be identified as the source of order reduction; the approximation of $g(t) = f(t, y(t))$ and the regularity of g are decisive [2,6].

The ABM predictor–corrector scheme

Uniform mesh: Let $t_n = nh$, $n = 0,1,\ldots,N$, $h = T/N$, and $f_n = f(t_n, y_n)$. On $[t_j, t_{j+1}]$ the function g is replaced by a polynomial interpolant and the kernel is integrated exactly. Piecewise constant interpolation ($g \approx f_j$) gives the fractional Adams–Bashforth predictor

$$ y_{n+1}^{P} = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f_j, \tag{4} $$

$$ b_{j,n+1} = \frac{h^{\alpha}}{\alpha} \left[(n+1-j)^{\alpha} - (n-j)^{\alpha} \right]. \tag{5} $$

The predictor is explicit. Piecewise linear interpolation ($g \approx f_j \frac{t_{j+1} - \tau}{h} + f_{j+1} \frac{\tau - t_j}{h}$) with f_{n+1} replaced by $f(t_{n+1}, y_{n+1}^{P})$ gives the Adams–Moulton corrector

$$ y_{n+1} = y_0 + \frac{h^{\alpha}}{\Gamma(\alpha+2)} \left[f\left(t_{n+1}, y_{n+1}^{P}\right) + \sum_{j=0}^{n} a_{j,n+1} f_j \right]. \tag{6} $$

Collecting the coefficient of f_j from the two adjacent subintervals $[t_{j-1}, t_j]$ and $[t_j, t_{j+1}]$ yields

$$a_{0,n+1} = n^{\alpha+1} - (n-\alpha)(n+1)^{\alpha}, \qquad (7)$$

$$a_{j,n+1} = (n-j+2)^{\alpha+1} - 2(n-j+1)^{\alpha+1} + (n-j)^{\alpha+1}, \qquad 1 \le j \le n, \qquad (8)$$

where $a_{0,n+1}$ receives a contribution from the first subinterval only, and the new node carries the unit coefficient $a_{n+1,n+1} = 1$, which is why $f(t_{n+1}, y_{n+1}^P)$ enters (6) with weight one. Formulas (4)–(8) are the fractional Adams predictor–corrector method of [1,2]. No nonlinear algebraic equation has to be solved: the implicit dependence on y_{n+1} is replaced by the predicted value. Further corrector iterations may be added if desired [6], but they are not used here.

Arbitrary (graded) mesh

For an arbitrary mesh $0 = t_0 < t_1 < \ldots < t_N = T$ with $h_j = t_{j+1} - t_j$, the same construction gives

$$y_{n+1}^P = y_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f_j, \qquad b_{j,n+1} = \frac{(t_{n+1} - t_j)^{\alpha} - (t_{n+1} - t_{j+1})^{\alpha}}{\alpha}, \qquad (9)$$

$$A_{j,n+1} = \frac{1}{h_j} \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} (t_{j+1} - \tau) d\tau, \qquad (10)$$

$$B_{j,n+1} = \frac{1}{h_j} \int_{t_j}^{t_{j+1}} (t_{n+1} - \tau)^{\alpha-1} (\tau - t_j) d\tau, \qquad (11)$$

$$y_{n+1} = y_0 + \frac{1}{\Gamma(\alpha)} \left[\sum_{j=0}^{n} \left(A_{j,n+1} f_j + B_{j,n+1} f_{j+1} \right) \right]_{f_{n+1} \to f(t_{n+1}, y_{n+1}^P)}.$$

On a uniform mesh, (9)-(12) reduce to (4)-(8). Closed-form expressions for $A_{j,n+1}$ and $B_{j,n+1}$ suffer from severe cancellation when h_j is very small, which happens near $t = 0$ on strongly graded meshes. In the implementation used here, the integrals are therefore evaluated in closed form only for $j = n$ (where $t_{n+1} - \tau$ ranges over $[0, h_n]$) and

by 12-point Gauss–Legendre quadrature for $j < n$, where the integrand is smooth on the subinterval. This implementation reproduces the uniform-mesh errors of Test 1 below to the printed precision (checked for $\alpha = 0.6$, $N = 40,80,160$). The graded mesh is

$$t_n = T\left(\frac{n}{N}\right)^r, \qquad n = 0,\ldots,N, \quad r \geq 1,$$

with $r = 1$ corresponding to the uniform mesh.

A frequently repeated shortcut is that linear interpolation in the corrector makes the complete ABM method second-order for every $0 < \alpha < 1$. This is not correct in general: the global order is determined by the complete predictor–corrector error propagation. The classical result is the following.

Theorem 1 (Diethelm, Ford, Freed [2]). Let $0 < \alpha < 1$, let f be continuous and Lipschitz continuous with respect to y on a bounded region containing the exact and the numerical solutions, and assume ${}^C D^\alpha y \in C^2[0,T]$. Then there is a constant C independent of h such that

$$\max_{0 \leq n \leq N} | y(t_n) - y_n | \leq C h^p, \qquad p = \min\{2, 1+\alpha\} = 1+\alpha.$$

Mechanism of the order limitation (sketch; the full proof with error propagation is given in [2]). Write $g(t) = f(t, y(t)) = {}^C D^\alpha y(t)$. If $g \in C^2[0,T]$, the piecewise constant rule of the predictor makes a quadrature error of size $O(h)$, so $y_{n+1}^P - y(t_{n+1}) = O(h)$. In the corrector this error is multiplied by $\dfrac{h^\alpha}{\Gamma(\alpha+2)}$ and by the Lipschitz constant, which gives a contribution $O(h^{1+\alpha})$. The piecewise linear rule of the corrector itself has quadrature error $O(h^2)$. The total local defect is therefore $O(h^{1+\alpha}) + O(h^2)$, and for $0 < \alpha < 1$ the first term dominates. The weak singularity of the kernel does not enter this argument, because it is integrated exactly. The hypothesis of Theorem 1 concerns $g = {}^C D^\alpha y$, which is stronger than $y \in C^2[0,T]$ and is not implied by it. For the manufactured solution $y(t) = t^2$ used below,

$$ {}^{C}D_{0+}^{\alpha} t^2 = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha}, \qquad (12) $$

$$ \frac{d^2}{dt^2} {}^{C}D_{0+}^{\alpha} t^2 = \frac{2(2-\alpha)(1-\alpha)}{\Gamma(3-\alpha)} t^{-\alpha}, \qquad (13) $$

and the second expression is unbounded at $t=0$ for $0<\alpha<1$. Hence, although $y(t)=t^2$ is smooth, Test 1 and Test 2 do not satisfy the hypothesis ${}^{C}D^{\alpha} y \in C^2[0,T]$ literally, and their results must be read as an empirical study under weaker regularity, not as a direct verification of Theorem 1. Since the theorem is an upper bound on the error, observed orders at or above p are consistent with it.

Lipschitz condition: The right-hand side used in Test 1 is $f(t,y) = -y^2 + t^4 + \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha}$. It is not globally Lipschitz in y, since $\partial f / \partial y = -2y$ is unbounded on □. On a bounded region $|y| \le M$ one has

$$ |f(t,u) - f(t,v)| \le 2M|u-v|, $$

so the bounded-region Lipschitz condition of Theorem 1 holds with $L = 2M$ (here the exact solution satisfies $0 \le y \le T^2$).

Perturbation stability. Let y_n and z_n be two sequences produced by the scheme (4)-(8) from initial values y_0 and z_0, with all values in a bounded region where f is Lipschitz in y with constant L, and put $e_n = y_n - z_n$. The predictor and corrector give $|e_{n+1}^{P}| \le |e_0| + \frac{Lh^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=0}^{n} \beta_{j,n+1} |e_j|$ with $\beta_{j,n+1} = (n+1-j)^{\alpha} - (n-j)^{\alpha} \ge 0$, and, since $a_{j,n+1} \ge 0$, inserting this into the corrector yields

$$ |e_{n+1}| \le \left(1 + \frac{Lh^{\alpha}}{\Gamma(\alpha+2)}\right) |e_0| + \sum_{j=0}^{n} \omega_{j,n+1} |e_j|, \qquad \omega_{j,n+1} = \frac{Lh^{\alpha}}{\Gamma(\alpha+2)} \left(a_{j,n+1} + \frac{Lh^{\alpha}}{\Gamma(\alpha+1)} \beta_{j,n+1} \right). \qquad (14) $$

The weights satisfy $0 \le \omega_{j,n+1} \le ch^{\alpha}(n+1-j)^{\alpha-1}$ with c depending only on α and L. The right-hand side of (19) contains only $e_0, \ldots, e_n$, because the predictor–corrector step

is explicit; hence no smallness condition on h is needed (such a condition, of the type $Lh^{\alpha}/\Gamma(\alpha+2)<1$, would arise only if the implicit corrector equation were solved exactly). The weakly singular *discrete* Gronwall inequality of Dixon and McKee [4] then gives

$$\max_{0\le n\le N}|y_n-z_n|\le C_T|y_0-z_0|, \quad (15)$$

with C_T depending on T, α and L but not on h. Additional perturbations in the evaluation of f (for example rounding errors) enter (19) as extra terms δ_{n+1} and are propagated in the same way. This is a bounded perturbation-stability statement (a sketch based on [4]); it is not an absolute-stability-region analysis, and no claim is made here about the stability region of the scheme or about its stability relative to other methods. The continuous Gronwall inequality of [11] does not apply directly to the discrete recursion, which is why a discrete version [4] is used.

All computations were done in IEEE double precision (Python/NumPy) with the direct history-sum implementation and a single correction step. The maximum nodal error and the experimental order of convergence are

$$E_N=\max_{0\le n\le N}|y(t_n)-y_n|,$$

$$\mathrm{EOC}_N=\log_2\left(\frac{E_N}{E_{2N}}\right).$$

For graded meshes the EOC is also taken with respect to $N\to 2N$. In Tables 1 and 2 the EOC for $N\to 2N$ is listed in the row of 2N; Table 4 labels the pair of meshes explicitly.

Nonlinear problem with exact solution $y=t^2$

On [0,1] consider

$${}^{C}D^{\alpha}_{0+}y(t)=-y(t)^2+t^4+\frac{2}{\Gamma(3-\alpha)}t^{2-\alpha}, \qquad y(0)=0. \quad (16)$$

By (16), $y(t)=t^2$ is the exact solution, since $-y^2+t^4=0$ along it. Table 1 gives the errors and EOC for $N=40,\ldots,5120$.

Table 1. Test 1 (nonlinear, f depends on y): maximum error E_N and EOC, uniform mesh.

N	$\alpha = 0.3$		$\alpha = 0.6$		$\alpha = 0.9$	
	E_N	**EOC**	E_N	**EOC**	E_N	**EOC**
40	1.680e-02	–	2.677e-03	–	7.322e-04	–
80	5.615e-03	1.581	8.242e-04	1.699	1.933e-04	1.921
160	1.934e-03	1.538	2.592e-04	1.669	5.135e-05	1.913
320	6.889e-04	1.489	8.277e-05	1.647	1.369e-05	1.908
640	2.525e-04	1.448	2.671e-05	1.632	3.655e-06	1.905
1280	9.465e-05	1.416	8.681e-06	1.621	9.769e-07	1.903
2560	3.608e-05	1.391	2.835e-06	1.614	2.613e-07	1.903
5120	1.394e-05	1.372	9.289e-07	1.610	6.989e-08	1.902

The EOC depends clearly on α and is below 2 for all three values. At $N = 5120$ it equals 1.372, 1.610 and 1.902, compared with $p = 1.3$, 1.6, 1.9, as they should for an upper bound. For $\alpha = 0.3$ the EOC is still decreasing at $N = 5120$ $(1.416 \to 1.391 \to 1.372)$, so it has not yet reached a limit; the numbers 1.37, 1.61, 1.90 are the last computed values, not asymptotic limits. As explained in Section 4, this test does not satisfy the regularity hypothesis of Theorem 1 literally.

Control problem isolating the predictor:

To test the mechanism of Section 4, the right-hand side is replaced by one that does not depend on y but has the same exact solution $y = t^2$ and the same function $g(t) = \dfrac{2}{\Gamma(3-\alpha)} t^{2-\alpha}$:

$$ {}^{C}D_{0+}^{\alpha} y(t) = \frac{2}{\Gamma(3-\alpha)} t^{2-\alpha}, \qquad y(0) = 0. $$

Here $f(t_{n+1}, y_{n+1}^P) = f(t_{n+1}, y_{n+1})$, so the predictor has no influence and the scheme reduces to the product trapezoidal rule for g. Regularity of g is identical to Test 1, and the kernel is the same.

Table 2. Test 2 (control, f independent of y): maximum error E_N and EOC, uniform mesh.

N	$\alpha = 0.3$		$\alpha = 0.6$		$\alpha = 0.9$	
	E_N	**EOC**	E_N	**EOC**	E_N	**EOC**
40	9.207e-05	–	9.817e-05	–	5.162e-05	–
80	2.360e-05	1.964	2.492e-05	1.978	1.378e-05	1.905
160	6.015e-06	1.972	6.302e-06	1.984	3.651e-06	1.917
320	1.527e-06	1.978	1.589e-06	1.988	9.607e-07	1.926
640	3.866e-07	1.982	3.996e-07	1.991	2.513e-07	1.934
1280	9.761e-08	1.986	1.004e-07	1.993	6.544e-08	1.941
2560	2.460e-08	1.989	2.517e-08	1.995	1.697e-08	1.947
5120	6.188e-09	1.991	6.309e-09	1.996	4.384e-09	1.953.

The EOC at $N = 5120$ is 1.991, 1.996 and 1.95, that is, close to 2 (for $\alpha = 0.9$ it is still increasing towards 2: $1.905 \to 1.953$). Tests 1 and 2 differ only in whether the predictor value influences the result. The drop from about 2 to about $1+\alpha$ therefore is caused by the predictor, and neither by the weakly singular kernel nor by the regularity of g, in agreement with the mechanism of Section 4.

The function $y(t) = t^\alpha$ has an unbounded derivative at $t = 0$, and ${}^C D^\alpha y \equiv \Gamma(\alpha+1)$. One might expect it to illustrate the loss of accuracy of uniform meshes. It does not: with

$${}^C D_{0+}^\alpha y(t) = \Gamma(\alpha+1), \qquad y(0) = 0,$$

the function $g \equiv \Gamma(\alpha+1)$ is constant, and product integration integrates constants exactly. Computationally the ABM solution coincides with t_n^α up to rounding: for $\alpha = 0.3, 0.6, 0.9$ and $N = 40, \ldots, 1280$ the maximum error was between 10^{-14} and 10^{-12}. Since the accuracy of the ABM scheme is governed by the regularity of $g = {}^C D^\alpha y$ and not of y, a meaningful singular test must have a non-smooth g; this is the case in Test 3.

Take $y(t) = t^{2\alpha}$. Then ${}^C D^\alpha y(t) = c_\alpha t^\alpha$ with $c_\alpha = \Gamma(2\alpha+1)/\Gamma(\alpha+1)$, so that $g(t) = c_\alpha t^\alpha$ is not differentiable at $t = 0$ with bounded derivative. The nonlinear problem is

$$ {}^C D_{0+}^\alpha y(t) = -y(t)^2 + t^{4\alpha} + c_\alpha t^\alpha, \qquad y(0) = 0, \quad t \in [0,1], \tag{17}$$

and it has the exact solution $y(t) = t^{2\alpha}$. It was solved on graded meshes (13) with $r = 1, 2, 3, 4$ and $N = 40, \ldots, 1280$. Tables 3 and 4 give the errors and the EOC; Table 5 summarises all three tests.

Table 3. Test 3 ($y = t^{2\alpha}$): maximum error E_N on the graded mesh (13); $r = 1$ is the uniform mesh.

α	N	$r=1$	$r=2$	$r=3$	$r=4$
0.3	40	1.265e-02	8.969e-03	1.751e-02	2.812e-02
	80	9.382e-03	2.646e-03	5.067e-03	8.238e-03
	160	6.557e-03	8.446e-04	1.552e-03	2.460e-03
	320	4.455e-03	2.873e-04	5.105e-04	7.877e-04
	640	2.985e-03	1.023e-04	1.772e-04	2.680e-04
	1280	1.986e-03	3.757e-05	6.395e-05	9.529e-05
0.6	40	1.632e-03	3.091e-03	5.923e-03	9.611e-03
	80	7.139e-04	9.231e-04	1.734e-03	2.782e-03
	160	3.112e-04	2.847e-04	5.253e-04	8.318e-04

α	N	$r=1$	$r=2$	$r=3$	$r=4$
	320	1.355e-04	8.982e-05	1.637e-04	2.564e-04
	640	5.898e-05	2.878e-05	5.198e-05	8.086e-05
	1280	2.568e-05	9.315e-06	1.673e-05	2.589e-05
0.9	40	5.807e-04	1.624e-03	3.267e-03	5.483e-03
	80	1.524e-04	4.248e-04	8.502e-04	1.424e-03
	160	4.032e-05	1.122e-04	2.236e-04	3.732e-04
	320	1.071e-05	2.984e-05	5.929e-05	9.870e-05
	640	2.852e-06	7.964e-06	1.579e-05	2.625e-05
	1280	7.608e-07	2.130e-06	4.219e-06	7.006e-06.

Table 4. Test 3: EOC for $N \to 2N$ (rows labelled by the coarser and finer N).

α	$N \to 2N$	$r=1$	$r=2$	$r=3$	$r=4$
0.3	40→80	0.43	1.76	1.79	1.77
	80→160	0.52	1.65	1.71	1.74
	160→320	0.56	1.56	1.60	1.64
	320→640	0.58	1.49	1.53	1.56
	640→1280	0.59	1.44	1.47	1.49
0.6	40→80	1.19	1.74	1.77	1.79
	80→160	1.20	1.70	1.72	1.74
	160→320	1.20	1.66	1.68	1.70
	320→640	1.20	1.64	1.65	1.67
	640→1280	1.20	1.63	1.64	1.64

α	$N \to 2N$	$r=1$	$r=2$	$r=3$	$r=4$
0.9	40→80	1.93	1.93	1.94	1.95.
	80→160	1.92	1.92	1.93	1.93
	160→320	1.91	1.91	1.92	1.92
	320→640	1.91	1.91	1.91	1.91
	640→1280	1.91	1.90	1.90	1.91

On the uniform mesh ($r=1$) the order deteriorates strongly for small α: the EOC is about 0.59 for $\alpha=0.3$ and 1.20 for $\alpha=0.6$ (close to 2α in both cases, which is noted here as an observation and not claimed as a general law), whereas for $\alpha=0.9$ no loss is visible in the tested range (EOC 1.91, r=2 changes this picture. At $N=1280$ the error for $\alpha=0.3$ falls from 1.99×10^{-3} to 3.76×10^{-5} (a factor of about $53\alpha=0.6$ from 2.57×10^{-5} to 9.31×10^{-6} (a factor of about 2.8, r=2 is 1.44 and 1.63 at the last refinement and is still decreasing; it approaches the predictor-limited order $1+\alpha$ gives no further gain in the order and larger errors, because the constants grow. For $\alpha=0.9$ grading does not help in these experiments: the uniform mesh is more accurate (for example 7.61×10^{-7} against 2.13×10^{-6} at $N=1280$ for $r=2$). Thus grading restores the order that is attainable for smooth g, namely $1+\alpha$, but it is not always beneficial, and the choice of r is problem dependent [5,9].

Table 5. Summary: EOC at the finest refinement (Tests 1 and 2: $N=2560\to5120$; Test 3: $N=640\to1280$).

α	$p=1+\alpha$	**Test 1**	**Test 2**	**Test 3,** $r=1$	**Test 3,** $r=2$
0.3	1.3	1.372	1.991	0.588	1.445
0.6	1.6	1.610	1.996	1.200	1.627
0.9	1.9	1.902	1.953	1.907	1.903

Test 3 uses the smaller maximal N; its values are therefore not directly comparable in the asymptotic sense with those of Tests 1 and 2.

Computational complexity. The direct implementation stores all history values f_j and evaluates a weighted sum over all previous levels at every step. At step n about n terms are processed, hence

$$\sum_{n=0}^{N-1} O(n) = O(N^2) \quad \text{operations}, \qquad O(N) \quad \text{memory}.$$

This persistent-memory cost is inherent to fractional problems [6]. On nonuniform meshes the constant is larger in our implementation because the weights $A_{j,n+1}$, $B_{j,n+1}$ depend on n and are computed by quadrature at every step (on a uniform mesh they depend only on $n-j$ and could be precomputed). Fast convolution or short-memory techniques can reduce the cost but are outside the scope of this paper. CPU times depend on the hardware and implementation and are not reported.

The results support four methodological points. First, the phrase "second-order ABM method" has to be used with care: the corrector contains a second-order component, but the complete algorithm has order $\min(2,1+\alpha)$ because of the first-order predictor (Test 2 shows order close to two once the predictor plays no role). Second, the weak singularity of the kernel is not by itself the cause of order loss, since product integration treats it analytically. Third, the regularity that governs accuracy is that of $g = {}^{C}D^{\alpha} y$; a smooth y (Test 1) and a singular y with constant g (Section 6.3) can both be misleading test cases. Fourth, graded meshes help when g is non-smooth at $t=0$ and α is small, but not uniformly in α.

Limitations of this study: only scalar equations with $0<\alpha<1$ are treated; only a single correction step is used; the stability result is a perturbation estimate and not a stability-region analysis; the optimal grading parameter was not determined, only $r=1,\ldots,4$ were compared; and only one non-smooth test problem was used.

Conclusions

The fractional ABM predictor–corrector method was derived from the Volterra integral equation for uniform and graded meshes. Under the classical hypothesis ${}^{C}D^{\alpha}y \in C^{2}[0,T]$ the uniform-mesh order is $p=\min(2,1+\alpha)=1+\alpha$ for $0<\alpha<1$, and an unconditional second-order claim is not justified. For the nonlinear manufactured problem with $y=t^{2}$ (which satisfies this hypothesis only in a weaker sense) the observed EOC at $N=5120$ is 1.372, 1.610, 1.902 for $\alpha=0.3,0.6,0.9$. A control problem with the same exact solution but y-independent right-hand side gives EOC close to 2, which identifies the predictor as the cause of the order loss. For a problem with non-smooth right-hand side along the solution, the uniform-mesh order drops markedly for small α, and grading with $r=2$ recovers order $1+\alpha$ and substantially reduces the error for $\alpha=0.3$ and 0.6. The perturbation stability of the scheme follows from a weakly singular discrete Gronwall inequality, and the direct implementation has cost $O(N^{2})$ in time and $O(N)$ in memory.

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