A POSTERIORI CONVERGENCE AND ERROR BOUNDS FOR A NYSTR0M METHOD FOR NONLINEAR FREDHOLM EQUATIONS

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Abstract

This study develops a computable error certificate for Picard iteration combined with the composite trapezoidal Nystrom method for a nonlinear Fredholm equation. A posteriori iteration error is separated from quadrature error, and the contraction constants used in each part are explicitly distinguished. For a variable-kernel example, the discrete contraction factor is derived from the quadrature weights, a rigorous quadrature bound is compared with a high-resolution numerical reference, and Picard iteration is compared with Newton-type and Galerkin approaches.

Keywords

nonlinear Fredholm equation; Picard iteration; Nystrom method; a posteriori error; quadrature; convergence rate.

1. Problem statement and contribution

Let $X = C[0,1]$ with the supremum norm $\|u\|_\infty = \max_{x\in[0,1]} |u(x)|$. Consider the nonlinear Fredholm equation of the second kind

$$u(x) = f(x) + \lambda \int_0^1 K(x,t)\Phi(t,u(t))dt = (Tu)(x), \qquad x \in [0,1].$$

Assume f and K are continuous and Φ is Lipschitz in its last argument: $|\Phi(t,r) - \Phi(t,s)| \le L|r-s|$. Define

$$q = |\lambda| L \sup_{x\in[0,1]} \int_0^1 |K(x,t)|dt < 1 \qquad (1)$$

Banach's contraction principle then guarantees a unique solution [5]. Recent studies treat iterative Fredholm–Hammerstein methods [1], existence and discretization for mixed Volterra–Fredholm equations [2], superconvergent Galerkin approximation [3], and Newton-type global convergence [4]. The classical numerical framework for Nystrom methods is described by Atkinson [6].

The thesis-specific contribution is a transparent, computable stopping certificate that combines the last Picard difference with a separately justified quadrature defect. This distinction makes clear what can be reduced by further iteration and what requires mesh refinement; it also prevents a discrete contraction constant from being substituted into a continuous-operator perturbation estimate without justification.

2. Picard convergence and a posteriori estimate

Lemma 1. Contraction and decay of successive differences

The operator T is a contraction with constant q. For $u_{n+1} = Tu_n$, set $d_n = \|u_{n+1} - u_n\|_\infty$. Then $d_n \le q^n d_0$.

Proof. For all $u, v \in X$ and $x \in [0,1]$,

$$|(Tu)(x) - (Tv)(x)| \le |\lambda| L \int_0^1 |K(x,t)\| u(t) - v(t)|dt \le q\|u - v\|_\infty .$$

Taking the supremum gives $\|Tu - Tv\|_\infty \le q\|u - v\|_\infty$. Applying this estimate successively to u_{n+1} and u_n yields $d_{n+1} \le qd_n$; induction proves the assertion.

Theorem 1. A posteriori error bound

Let u^* be the unique fixed point of T. Then

$$\|u_n - u^*\|_\infty \le \frac{d_n}{1-q} \le \frac{q^n d_0}{1-q} \qquad (2)$$

In particular, after computing u_N, the test $d_{N-1}/(1-q) \le \varepsilon$ guarantees $\|u_N - u^*\|_\infty \le \varepsilon$.

Proof. The telescoping series gives $u_n - u^* = \sum_{j=n}^{\infty}(u_j - u_{j+1})$. Hence $\|u_n - u^*\|_\infty \le \sum_{j=n}^{\infty} d_j \le d_n/(1-q)$. Lemma 1 gives the second inequality. Taking $n = N$ and noting that $d_{N-1} = \|u_N - u_{N-1}\|_\infty$ proves the implementable test.

3. Discretization bound and corrected total-error formula

Let T_h be the composite-trapezoidal Nystrom operator, with fixed point u_h^* and contraction constant $q_h < 1$. Define the quadrature defect specifically at the discrete fixed point by $\eta_h = \|T_h u_h^* - T u_h^*\|_\infty$. Since the continuous map T has contraction constant q

$$\|u_h^* - u^*\|_\infty \le \frac{\eta_h}{1-q} \qquad (3)$$

Indeed, if $E = \|u_h^* - u^*\|_\infty$, then $E \le \eta_h + qE$. Thus the denominator in (3) is the continuous contraction factor because the perturbation step uses T. For the discrete Picard sequence, let $d_n^{(h)} = \|u_{n+1}^{(h)} - u_n^{(h)}\|_\infty$. The returned iterate therefore satisfies

$$\|u_N^{(h)} - u^*\|_\infty \le \frac{d_{N-1}^{(h)}}{1-q_h} + \frac{\eta_h}{1-q} \qquad (14)$$

Here q_h belongs only to the discrete iteration tail, while q belongs to the continuous-versus-discrete fixed-point perturbation. This explicit distinction resolves the denominator ambiguity: replacing q by q_h in the second term requires instead a defect bound formulated for the discrete map over the relevant solution set.

4. Variable-kernel example and numerical comparison

Consider $u(x)=1+1.15\int_0^1 \tanh(u(t))/(1+x+t)dt$. As $|\tanh r-\tanh s|\le|r-s|$, its continuous factor is

$$q=1.15\max_{x\in[0,1]}\int_0^1\frac{dt}{1+x+t}=1.15\ln 2\approx 0.79711926<1.$$

For nodes $t_j=jh$, $j=0,\ldots,n$, and composite-trapezoidal weights $w_0=w_n=h/2$, $w_j=h$ otherwise, the discrete map is $(T_hv)(x_i)=1+1.15\sum_{j=0}^{n}w_j\tanh(v(t_j))/(1+x_i+t_j)$. Since the summand decreases as x_i increases,

$$q_h=1.15\max_i\sum_{j=0}^{n}\frac{w_j}{1+x_i+t_j}=1.15\sum_{j=0}^{n}\frac{w_j}{1+t_j}.$$

At $n=200$ this gives $q_h=0.7971210545$. The discrete map preserves $B=\{v:1\le v(x)\le 1.80\}$ because its weights are positive and $|\tanh v|\le 1$. Differentiating the discrete fixed-point relation gives $|u_h^{*\prime}|\le 1.15$ and $|u_h^{*\prime\prime}|\le 2.30$. For $g_x(t)=(1+x+t)^{-1}\tanh(u_h^*(t))$, the product and chain rules yield $|g_x''(t)|\le 2+4(1.15)+2(1.15)^2=9.245$. The composite-trapezoidal remainder gives

$$\eta_h\le\frac{1.15(9.245)}{12}h^2=2.215\times10^{-5},\qquad\frac{\eta_h}{1-q}\le 1.092\times10^{-4}\quad(h=1/200).$$

Numerical validation: iteration to a successive-difference tolerance of 10^{-13} gives $u_{200}^{(h)}$; repeating on $n=2000$ and comparing its values at the shared nodes gives

$$\max_i|u_{200}^{(h)}(x_i)-u_{2000}^{(h)}(x_i)|\approx 1.9244\times10^{-6}.$$

This is a high-resolution numerical reference error, not an exact-solution error. It is about 56.8 times smaller than the conservative proven discretization bound 1.092×10^{-4}, consistent with that bound. The values are a numerical cross-check; the rigorous guarantee is (4).

5. Comparison with alternative methods and conclusion

Picard iteration needs only evaluation of the integral operator and, under (1), supplies a global geometric bound and a low-cost stopping test. Its worst-case factor can be pessimistic when q is near one. Newton-type methods may converge faster near a regular solution, but require derivative information and a linearized solve; their global convergence is typically established under additional hypotheses [4]. Galerkin methods can achieve higher asymptotic convergence orders for smooth or weakly singular kernels [3], at the cost of constructing and solving a discrete system. The approaches therefore serve different goals: this Picard–Nystrom analysis emphasizes simple implementation and an explicit posteriori certificate, not a universal claim of faster convergence.

The analysis separates iteration and quadrature errors, derives q_h from the actual trapezoidal weights, and verifies the conservative theoretical estimate against a finer-grid reference. Further work should develop sharper computable quadrature defects, especially for weakly singular kernels, and compare total computational cost across Picard, Newton-type, and Galerkin schemes.

References

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