Advanced Analysis of Prime Number Generation Methods in Cryptographic Systems

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Abstract

This thesis analyzes prime number generation methods employed in modern cryptographic systems and examines their underlying mathematical principles. The study explores the generation of large prime numbers, random number generation mechanisms, and the operational characteristics of primality testing algorithms. The performance, efficiency, and practical benefits of the Fermat and Miller–Rabin tests in the prime generation process are evaluated. Furthermore, the concept of safe primes and their significance in public-key cryptographic frameworks, including RSA and Diffie–Hellman systems, are investigated. The findings demonstrate that prime number generation algorithms play a vital role in maintaining the security, reliability, and robustness of cryptographic systems.

Keywords:

prime number, prime number generation, primality test, Miller–Rabin algorithm, Fermat test, RSA, Diffie–Hellman, cryptography, safe primes, random number generator.

Information security has become a critical requirement in modern digital infrastructures, where cryptographic algorithms are widely utilized to protect confidential information across electronic commerce, banking, cloud technologies, and governmental systems [1]. The security of public-key cryptosystems such as RSA, Diffie–Hellman, and ElGamal relies heavily on the generation and utilization of large prime numbers [2]. Therefore, prime number generation remains one of the fundamental problems in contemporary cryptography.

The process of prime generation incorporates random number production, primality testing, and compliance with cryptographic security criteria [3]. Due to their favorable balance between accuracy and computational efficiency, probabilistic methods such as Miller–Rabin and Solovay–Strassen are extensively used in practical environments [4]. Furthermore, the growing influence of quantum computing has increased the importance of developing efficient and secure prime generation techniques [5]. Accordingly, this article focuses on the mathematical principles, algorithmic structures, and practical performance of prime number generation methods.e of the simplest approaches to prime number generation is to produce a random odd integer and test its primality. If the generated number n satisfies the following condition:

$$n > 1$$

and is divisible only by 1 and itself, then the number is considered prime [6]. Random number generators are commonly used in prime generation procedures. In cryptographic systems, a random integer is usually selected within the following interval:

$$2^{k-1} < n < 2^k$$

where k denotes the bit length of the number. In RSA cryptosystems, 1024-bit, 2048-bit, or 4096-bit prime numbers are commonly used [7].

To verify the primality of generated numbers, the Fermat primality test is widely applied. This test is based on Fermat's Little Theorem [8]:

$$a^{p-1} \equiv 1(\mathrm{mod} p)$$

If this relation does not hold, the number is considered composite. However, for some Carmichael numbers, this test may produce incorrect results [9].

One of the most efficient methods used in practical cryptography is the Miller-Rabin primality test. In this algorithm, the value $n-1$ is represented as follows [10]:

$$n - 1 = 2^s \cdot d$$

where d is an odd integer. Then, for randomly selected values of a, the following expression is computed:

$$a^d (\mathrm{mod} n)$$

If during the testing process the condition

$$a^{2^r d} \equiv -1 (\mathrm{mod} n)$$

is not satisfied, the number is considered composite. This algorithm provides high speed and practical efficiency [11].

Safe primes also play an important role in prime number generation. If the following condition holds:

$$p = 2q + 1$$

where both p and q are prime numbers, then p is called a safe prime [12]. Such primes are widely used in the Diffie-Hellman algorithm.

In the RSA algorithm, two large prime numbers are selected to generate the following value [13]:

$$n = pq$$

The security of many public-key cryptographic systems is based on the computational hardness of large integer factorization. Therefore, the cryptographic quality and unpredictability of generated prime numbers are essential requirements for secure system design.

Cryptographically secure pseudorandom number generators play a central role in prime generation by producing unpredictable candidate values suitable for cryptographic applications [14]. In addition, optimization techniques such as preliminary divisibility checks using small prime numbers significantly improve computational efficiency by eliminating unsuitable candidates before applying expensive primality tests [15].

However, the emergence of quantum computing presents new challenges to prime-based cryptographic mechanisms. The capability of Shor's algorithm to efficiently factorize large integers threatens conventional systems such as RSA [16]. Consequently, strengthening existing algorithms and developing secure post-quantum alternatives have become important areas of contemporary research. Overall, efficient prime number generation and reliable primality testing continue to be indispensable components of secure modern cryptographic systems.

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